Accepted Manuscript : https://doi.org/10.1016/j.ejor.2021.10.018

# A fully polynomial time approximation scheme for the probability maximizing shortest path problem

Jisun Lee[a], Seulgi Joung[c,*], Kyungsik Lee[b,*]

[a]*Department of Industrial Engineering and Operations Research, University of California, Berkeley, CA 94720, USA*
[b]*Department of Industrial Engineering, Seoul National University, 1 Gwanak-ro, Gwanak-gu, Seoul 08826, Republic of Korea*
[c]*Department of Industrial Engineering, Chonnam National University, 77, Yongbong-ro, Buk-gu, Gwangju 61186, Republic of Korea*

**Abstract**

In this paper, we consider a probability maximizing shortest path problem. For a given directed graph with the length of each arc being an independent normal random variable with rational mean and standard deviation, the problem is to find a simple *s*-*t* path that maximizes the probability of arriving at the destination within a given limit. We first prove that the problem is $\mathcal{NP}$-hard even on directed acyclic graphs with the mean of the length of each arc being restricted to an integer. Then, we present pseudo-polynomial time exact algorithms for the problem along with nontrivial special cases that can be solved in polynomial time. Finally, we present a fully polynomial approximation scheme (FPTAS) that iteratively solves deterministic shortest path problems. The structure of the proposed approximation scheme can be applied to devise FPTAS for other probability maximizing combinatorial optimization problems once the corresponding deterministic problems are polynomially solvable.

*Keywords:* Combinatorial optimization, Shortest path, Probability maximization, Exact algorithm, Fully polynomial time approximation scheme

*Corresponding authors
*Email addresses:* jisun_lee@berkeley.edu (Jisun Lee), sgjoung@jnu.ac.kr (Seulgi Joung), optima@snu.ac.kr (Kyungsik Lee)

## 1. Introduction

The shortest path problem is one of the fundamental combinatorial optimization problems. Given a directed graph $G = (V, A)$ with $|V| = n$ and $|A| = m$, a source node $s \in V$, a destination node $t \in V$, and the length (or travel time) $l_a$ of each arc $a \in A$, the problem is to find a minimum-length simple path from node $s$ to node $t$. It is well-known that the problem can be solved in $O(mn)$ if there is no negative cycle (Bellman, 1958). Further, if $l_a \geq 0$ for all $a \in A$, Dijkstra's algorithm runs in $O(n^2)$ (Dijkstra, 1959). Many studies on efficient algorithms for the shortest path problem and its variants have been carried out for decades (see, for example, Dantzig, 1960; Henig, 1986; Fredman & Tarjan, 1987; Ahuja et al., 1990; Cherkassky et al., 1996; Taccari., 2016). Among them, Fredman & Tarjan (1987) improved the time complexity of the Dijkstra's algorithm to $O(n \log n + m)$ by using a Fibonacci heap. In recent years, stochastic shortest path problems considering the uncertainty in travel time of each arc have been being actively studied by many researches including Mirchandani (1976), Sigal et al. (1980), Murthy & Sarkar (1997), Azaron & Kianfar (2003), Korkmaz & Krunz (2003), Nie & Wu (2009), Nikolova (2009), Xiao et al. (2011), Olya et al. (2014), Chassein & Goerigk (2015), Cheng & Lisser (2015), Wang et al. (2016), Chen et al. (2017), Conde (2017), Guillot & Stauffer (2020), Raith et al. (2018), Chassein et al. (2019), Duque & Medaglia (2019), Halman et al. (2019), Goldberg & Poss (2020), and Wang et al. (2020).

In this study, we consider the probability maximizing shortest path problem (PSP). The problem is to find a simple $s$-$t$ path that maximizes the probability of arriving at the destination within a given time budget $b$, when the travel time $l_a$ follows an independent normal distribution $\mathcal{N}(\mu_a, \sigma_a^2)$. To avoid confusion, we define $I_A = \{1, 2, \ldots, m\}$ as the index set of arcs, and $a(i)$ as the corresponding arc to index $i \in I_A$. Throughout the paper, we assume that $b \in \mathbb{Q}_+$, $\boldsymbol{\mu} \in \mathbb{Q}_+^m$, and $\boldsymbol{\sigma^2} \in \mathbb{Q}_+^m$ unless otherwise specified, where $\boldsymbol{\mu} := (\mu_{a(1)}, \ldots, \mu_{a(m)})$ is the vector of means of arcs and $\boldsymbol{\sigma^2} := (\sigma^2_{a(1)}, \ldots, \sigma^2_{a(m)})$ is that of variances of arcs. PSP has practical applications such as routing problems in the context of transportation (Nie & Wu, 2009; Chen et al., 2017) and telecommunication (Korkmaz & Krunz, 2003). It is also closely related to the product-sum path problem (PSPP) considering the survival probability which can be applied to find safe and short paths in a city (Galbrun et al., 2016; Halman et al., 2019). That is, PSP is equivalent to a variation of PSPP of Halman et al. (2019), if $l_a$ for each arc $a \in A$ follows the distribution, $Pr(l_a = \mu_a) = p_a$ and $Pr(l_a = \infty) = 1 - p_a$, where $p_a$ is the given survival probability of the arc.

Let $\mathcal{P}$ be the set of all simple $s$-$t$ paths on $G$, and let us represent a simple $s$-$t$ path in $\mathcal{P}$ by its corresponding subset of arcs $P \subseteq A$. For the ease of later expositions, we also represent a path $P \in \mathcal{P}$ by its characteristic vector $x^P \in \mathbb{B}^m$, that is, $x_a^P = 1$ if $a \in P$ and $x_a^P = 0$ otherwise, and let $X_{st} \subseteq \mathbb{B}^m$ be the set of all possible characteristic vectors of $s$-$t$ paths in $\mathcal{P}$. For a given path $P \in \mathcal{P}$, if we let $l(P) := \sum_{a \in P} l_a$, then PSP is defined as

$$\rho^* = \max_{P \in \mathcal{P}} Pr\left(l(P) \leq b\right),$$

which is the same problem as

$$\max_{x \in X_{st}} Pr\left(\sum_{a \in A} l_a x_a \leq b\right).$$

Since we assume that the length of each arc follows an independent normal distribution, PSP is equivalent to the following problem, which we call the $Z$-value problem (ZP in short).

$$z^* = \max_{P \in \mathcal{P}} (b - \mu(P))/\sigma(P),$$

where $\mu(P) := \sum_{a \in P} \mu_a$ and $\sigma(P) := \sqrt{\sum_{a \in P} \sigma_a^2}$. Note that, while the optimal solutions of PSP and ZP are equivalent in that they have the same probability of arriving at the destination within the given time budget $b$, their approximate solutions with the same relative error are not necessarily equivalent. However, Nikolova (2009) proved that an approximate solution of ZP is also an approximate solution of PSP with the same relative error if there exists a path $P \in \mathcal{P}$ such that $\mu(P) \leq b$. PSP and ZP can be classified into two special cases as follows:

- Case 1: There exists a path $P \in \mathcal{P}$ such that $\mu(P) \leq b$. For this case, which we call PSP-1 (or ZP-1), $\rho^* \geq 0.5$ and $z^* \geq 0$.
- Case 2: For each path $P \in \mathcal{P}$, $\mu(P) > b$. For this case, which we call PSP-2 (or ZP-2), $\rho^* < 0.5$ and $z^* < 0$.

Furthermore, if a problem is defined on a directed acyclic graph, we append '-DAG' to the name of the problem. Otherwise, it means that the problem is defined on a general directed graph. For example, PSP-2-DAG is PSP-2 defined on a directed acyclic graph (DAG). Note that, for a given instance of PSP, checking whether the instance belongs to PSP-1 or PSP-2 can be done in polynomial time by solving a deterministic shortest path problem.

Xiao et al. (2011) proved that PSP-2 is strongly $\mathcal{NP}$-hard by showing that the *longest path problem*, which is strongly $\mathcal{NP}$-hard (Johnson & Garey, 1979), can be polynomially transformed to PSP-2. The theoretical complexity of PSP-1, as far as we know, has not been settled down, but Nikolova et al. (2006)

showed that an upper bound on the computational complexity of exact algorithms for PSP-1 is $O(n^{\log n})$ based on the relation with the parametric shortest path problem.

For exact algorithms to solve PSP, Nie & Wu (2009) suggested a label-correcting algorithm which runs in $O(mn^{2n-1}b + mn^n b^2)$. Chen et al. (2017) proposed an $O(mp^2 + m + n \log n)$ algorithm by enumerating all non-dominated paths, where $p = |\mathcal{P}|$. On the other hand, Nikolova (2009) proposed a pseudo-polynomial time exact algorithm for ZP that runs in $O(nm \max_{a \in A} \sigma_a^2)$ when $\sigma_a^2$ values are integral, but this algorithm does not guarantee that the obtained solution is a simple path.

Approximation approaches have been studied as well. Nikolova (2009) devised a fully polynomial time approximation scheme (FPTAS) for ZP-1 whose complexity is $O(\log(\sigma_{\max}^2/\sigma_{\min}^2) \log(f_u/f_l)(1/\epsilon^2)n^2)$ for any given approximation ratio $0 < \epsilon < 1$, where $\sigma_{\max} = \max_{P \in \mathcal{P}} \sigma(P)$, $\sigma_{\min} = \min_{P \in \mathcal{P}} \sigma(P)$, and $f_u$ and $f_l$ are upper and lower bounds on $(b - \mu(P))/\sigma(P)$ for all $P \in \mathcal{P}$, respectively. The study also proved that an approximation for ZP-1 with a relative error $\epsilon$ yields an approximation for PSP-1 with the same relative error in Lemma 3.4.1 of the paper. Therefore, the proposed FPTAS for ZP-1 serves also as an FPTAS for PSP-1. However, an approximation for ZP-2 with a relative error $\epsilon$ does not necessarily yield an approximation for PSP-2 with the same relative error: Consider an instance of ZP-2 with $z^* = -1$. Suppose that we have a solution of the instance whose objective function value $z^\epsilon = -1.1$. The solution is an approximate solution of ZP-2 with a relative error $\epsilon = 0.1$. However, it is not an approximate solution of PSP-2 with the same relative error because $\Phi(z^\epsilon) < (1 - \epsilon) \cdot \Phi(z^*)$, where $\Phi(\cdot)$ is the standard normal (cumulative) distribution function (Recall that PSP-2 is $\mathcal{NP}$-hard in the strong sense, so the existence of an FPTAS for it is unlikely). Recently, Xiao et al. (2011) proposed an FPTAS for PSP-1 which runs in $O((1/\epsilon) \cdot m^2 n \log n)$ for any given approximation ratio $0 < \epsilon < 1$.

Table 1 summarizes the previous research results mentioned above together with our contributions given as follows.

- As mentioned above, PSP, more specifically PSP-2, has been shown to be strongly $\mathcal{NP}$-hard (Xiao et al., 2011) based on the fact that the longest path problem is strongly $\mathcal{NP}$-hard in general. However, the longest path problem is polynomially solvable on directed acyclic graphs, so that the proof by Xiao et al. (2011) does not mean that PSP-2-DAG is strongly $\mathcal{NP}$-hard. In this paper, we show for the first time that PSP-2-DAG is $\mathcal{NP}$-hard with $\boldsymbol{\mu} \in \mathbb{Z}_+^m$ (Proposition 1), but it can be solved in pseudo-polynomial time (Proposition 3). A pseudo-polynomial time exact algorithm for PSP-1 defined on any graphs with $\boldsymbol{\mu} \in \mathbb{Z}_+^m$ is also proposed (Proposition 2). Furthermore, we show that there exist

Table 1: Summary of results for PSP and ZP

| Reference | $\mathcal{NP}$-hardness | Exact algorithm | | | FPTAS | |
|---|---|---|---|---|---|---|
| | Result | Problem | Restriction | Complexity | Problem | Complexity |
| Nikolova et al. (2006)* | - | PSP-1 | none | exponential | - | - |
| Nie & Wu (2009)* | - | PSP | $b \in \mathbb{Z}_+$ | exponential | - | - |
| Nikolova (2009)* | - | ZP | $\sigma^2 \in \mathbb{Z}_+^m$ | pseudo-polynomial‡ | ZP-1 | quadratic in $1/\epsilon$ |
| Xiao et al. (2011)* | PSP-2: strongly $\mathcal{NP}$-hard⋆ | - | - | - | PSP-1 | linear in $1/\epsilon$ |
| Chen et al. (2017)* | - | PSP | $\mu \in \mathbb{Z}_+^m$ | exponential | - | - |
| Our work | PSP-2-DAG: $\mathcal{NP}$-hard | PSP-1 | $\mu \in \mathbb{Z}_+^m$ | pseudo-polynomial | PSP-1 | sublinear in $1/\epsilon$ |
| | | PSP-2-DAG | $\mu \in \mathbb{Z}_+^m$ | pseudo-polynomial | | |
| | | PSP-1 | special case† | polynomial | | |

(*) : The previous studies did not clarify whether the input data type is rational or not. However, it may well be presumed that the input data type is rational because they implicitly assume that each elementary arithmetic operation can be done in $O(1)$.

(⋆) : It does not mean that PSP-2-DAG is $\mathcal{NP}$-hard.

(†) : Each arc can be classified into one of $p$ classes according to its variance for a fixed positive integer $p$.

(‡) : A solution obtained by the algorithm is not guaranteed to be a simple path.

non-trivial special cases of PSP-1 that can be solved in polynomial time (Proposition 4).

- The FPTAS proposed by Nikolova (2009) for ZP-1 can be readily adapted to any combinatorial optimization problem that is polynomially solvable since the scheme iteratively solves the deterministic shortest path problem as a subroutine. They approximated the nonlinear level sets of the objective functions by multiple linear segments. However, the running time of the scheme is not *strongly polynomial*, that is, it depends on the magnitude of numeric values of input data because the required numbers of linear segments and level sets are not strongly polynomial. On the other hand, the complexity of the FPTAS proposed by Xiao et al. (2011) is strongly polynomial, but their scheme is specially devised for PSP-1 so that it can not be readily adapted to other combinatorial optimization problems. In this paper, we propose a strongly polynomial FPTAS for PSP-1 whose time complexity is $O((m + n \log n) \cdot m^{1.5} \sqrt{1/\epsilon} \log(1/\epsilon))$ (Theorem 2). Its running time is faster than the FPTAS proposed by Xiao et al. (2011).

- Another notable contribution of the proposed FPTAS is that its running time is sublinear in $1/\epsilon$. Due to this property, the running time of the proposed FPTAS grows more slowly, compared to that of the FPTAS proposed by Xiao et al. (2011), as $\epsilon$ gets smaller (Figure 5). In a broader context, there

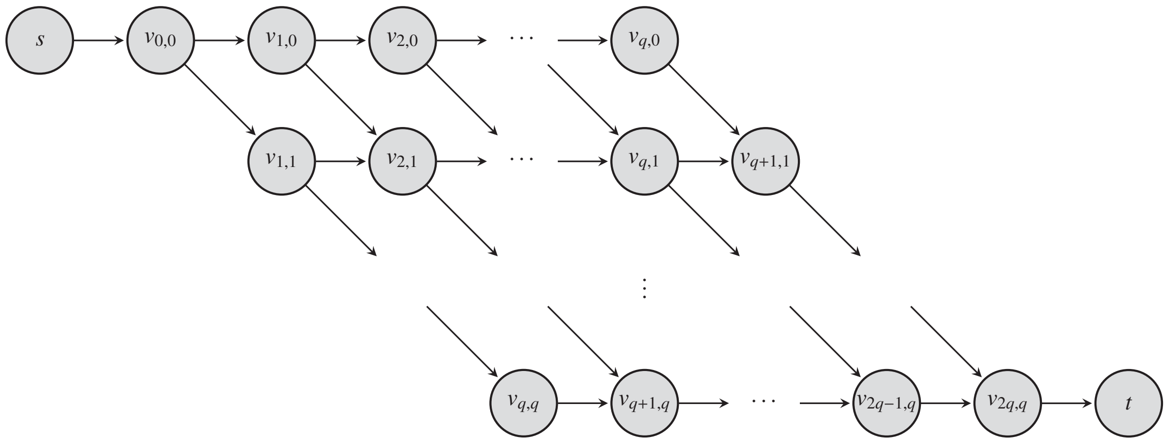



Figure 1: Directed acyclic graph for the proof of Proposition 1

exists an FPTAS whose running time is sublinearly dependent on $1/\epsilon$ (e.g., Knauer et al. (2018) for an elastic matching problem in computational geometry). However, to the best of our knowledge, it is difficult to find out a study in the existing literature on combinatorial optimization problems (not limited to PSP) that proposed an FPTAS whose running time is sublinear in $1/\epsilon$. In addition, the structure of the proposed FPTAS is generic so that it can be adapted to any probability maximizing combinatorial optimization problems once the corresponding deterministic problem is polynomially solvable (Theorem 3).

The remainder of this paper is organized as follows. In Section 2, we analyze the computational complexity of PSP-2-DAG, and then present exact algorithms for PSP-1 and PSP-2-DAG along with polynomially solvable special cases of PSP-1. In Section 3, an FPTAS for PSP-1 is presented. Finally, concluding remarks with some future research directions are given in Section 4.

## 2. Computational Complexity and Exact Algorithms

In this section, we first show that PSP-2-DAG is $\mathcal{NP}$-hard with $\mu_a \in \mathbb{Z}_+$ for all $a \in A$.

**Proposition 1.** *PSP-2-DAG with $\boldsymbol{\mu} \in \mathbb{Z}_+^m$ is $\mathcal{NP}$-hard.*

*Proof.* We show that every instance of the Equal Cardinality Partition (ECP) problem, which is known to be $\mathcal{NP}$-complete (Karp, 1972; Johnson & Garey, 1979), can be solved by solving an instance of PSP-2-DAG with $\mu_a \in \mathbb{Z}_+$ for all $a \in A$. An instance of ECP consists of a finite set $W = \{1, \ldots, 2q\}$ with a size $w_k \in \mathbb{Z}_+$

for each $k \in W$ and a bound $C \in \mathbb{Z}_+$ such that $\sum_{k \in W} w_k = 2C$. The problem is to determine if $W$ can be partitioned into two disjoint sets $W_1$ and $W_2$ such that $\sum_{k \in W_1} w_k = \sum_{k \in W_2} w_k = C$ and $|W_1| = |W_2| = q$.

Now, we construct the corresponding instance of PSP-2-DAG as follows. First, we construct a directed acyclic graph $G = (V, A)$ with $q^2 + 2q + 3$ nodes and $2q^2 + 2q + 2$ arcs as shown in Figure 1. Here, we define $V = \{s, t\} \cup \{v_{i,j} : 0 \le j \le q, j \le i \le j + q\}$ and $A = \{(s, v_{0,0}), (v_{2q,q}, t)\} \cup A_1 \cup A_2$, where $A_1 = \{(v_{i,j}, v_{i+1,j}) : 0 \le j \le q, j \le i \le j + q - 1\}, A_2 = \{(v_{i,j}, v_{i+1,j+1}) : 0 \le j \le q - 1, j \le i \le j + q\}$. For each arc $a = (v_{i,j}, v_{i+1,j}) \in A_1$, we set $\mu_a = w_{i+1}$ and $\sigma_a^2 = w_{i+1}$. For each arc $a = (v_{i,j}, v_{i+1,j+1}) \in A_2$, we set $\mu_a = C$ and $\sigma_a^2 = 0$. Finally, we set $\mu_{(s,v_{0,0})} = \mu_{(v_{2q,q},t)} = 0$, $\sigma^2_{(s,v_{0,0})} = \sigma^2_{(v_{2q,q},t)} = 0$, and $b = C(q - 1)$.

Obviously, exactly $q$ arcs of $A_1$ and $q$ arcs of $A_2$ will be chosen for any $s$-$t$ path of $G$, that is, $\sum_{a \in A_1} x_a = \sum_{a \in A_2} x_a = q$ and $\sum_{a \in A_2} \mu_a x_a = Cq$ for all $x \in X_{st}$. Therefore, there is no $s$-$t$ path $P \in \mathcal{P}$ such that $\mu(P) \le b$. By noting that $\mu_a = \sigma_a^2$ for all $a \in A_1$, we can see that optimal solutions to the instance of PSP-2-DAG maximize the objective value of

$$\frac{C(q-1) - \sum_{a \in A_1} \mu_a x_a - \sum_{a \in A_2} \mu_a x_a}{\sqrt{\sum_{a \in A_1} \sigma_a^2 x_a}} = \frac{-C - \sum_{a \in A_1} \mu_a x_a}{\sqrt{\sum_{a \in A_1} \mu_a x_a}}.$$

Let $f(X) = (-C - X)/\sqrt{X}$ for $X > 0$. Then, $f(X)$ is a quasi-concave function on $X$ and it attains the maximum value at the only point $X = C$. From the construction of the instance of PSP-2-DAG, we can readily see that the answer to the given instance of ECP is positive if and only if the obtained optimal solution $x^*$ satisfies $\sum_{a \in A_1} \mu_a x_a^* = C$. Therefore, the result follows. □

As mentioned in section 1, Xiao et al. (2011) already showed that PSP-2 is strongly $\mathcal{NP}$-hard in general. However, Proposition 1 shows that PSP-2 is still $\mathcal{NP}$-hard even on directed acyclic graphs with $\boldsymbol{\mu} \in \mathbb{Z}_+^m$, but it might not be strongly $\mathcal{NP}$-hard. Indeed, we show in Proposition 3 below that it can be solved in pseudo-polynomial time. We also show in Proposition 2 below that PSP-1 defined on any graphs with $\boldsymbol{\mu} \in \mathbb{Z}_+^m$ can be solved in pseudo-polynomial time.

**Proposition 2.** *An optimal solution to PSP-1 defined on a directed graph (not necessarily acyclic) with $b \in \mathbb{Q}_+$, $\boldsymbol{\mu} \in \mathbb{Z}_+^m$, and $\boldsymbol{\sigma^2} \in \mathbb{Q}_+^m$ can be obtained in $O(mn\lfloor b \rfloor)$ pseudo-polynomial time.*

*Proof.* Consider the following subproblem defined for each $k \in \{0, \ldots, \lfloor b \rfloor\}$, which is a special case of the resource constrained shortest path problem (Beasley & Christofides, 1989).

$$z_k = \min \left\{ \sum_{a \in A} \sigma_a^2 x_a : \sum_{a \in A} \mu_a x_a \le k, x \in X_{st} \right\}, \tag{1}$$

where $z_k = \infty$ if the subproblem is infeasible. Let $\hat{z} = \max\{(b-k)/\sqrt{z_k} : k = 0, \ldots, \lfloor b \rfloor \text{ such that } z_k < \infty\}$, and let $\hat{x} \in X_{st}$ be the corresponding solution. We claim that $\Phi(\hat{z})$ and $\hat{x}$ are the optimal value of PSP-1 and the corresponding optimal solution, respectively, where $\Phi(\cdot)$ is the cumulative distribution function of the standard normal random variable. Suppose that we are given an optimal solution $x^*$, but there is no $k$ such that $x^*$ is optimal to the corresponding subproblem (1). Then, it means $\hat{k} = \sum_{a \in A} \mu_a x_a^* \leq b$ and $\sum_{a \in A} \sigma_a^2 x_a^* > z_{\hat{k}}$ since we assume that there is no $k$ such that $x^*$ is optimal to $z_k$. It contradicts the optimality of $x^*$ since $(b-\hat{k})/\sqrt{z_{\hat{k}}} > (b-\hat{k})/\sqrt{\sum_{a \in A} \sigma_a^2 x_a^*}$. Therefore, PSP-1 with $\boldsymbol{\mu} \in \mathbb{Z}_+^m$ can be solved by solving subproblems (1) for all $k \in \{0, \ldots, \lfloor b \rfloor\}$.

For each $k \in \{0, \ldots, \lfloor b \rfloor\}$, the subproblem (1) can be solved in $O(mnk)$ by applying the dynamic programming (DP) algorithm of Beasley & Christofides (1989). Their DP algorithm guarantees that the obtained solution is a simple path because there is no negative cycle. If we apply the DP algorithm of Beasley & Christofides (1989) to the subproblem for some $k$, it actually solves the subproblems (1) for all $\bar{k}$ such that $0 \leq \bar{k} \leq k$. It means that we can get $z_k$ and the corresponding solution for all $k \in \{0, \ldots, \lfloor b \rfloor\}$ by applying their DP algorithm to the subproblem (1) for $k = \lfloor b \rfloor$. Then, to find $\hat{k}$ such that $(b-\hat{k})/\sqrt{z_{\hat{k}}}$ is the maximum, we compare $(b-k)^2/z_k$ values to avoid the calculations of square roots, which can be done in $O(\lfloor b \rfloor)$. Therefore, an optimal solution to PSP-1 with $\boldsymbol{\mu} \in \mathbb{Z}_+^m$ can be obtained in $O(mn\lfloor b \rfloor)$. □

Based on a similar idea to the case of PSP-1, we can devise a pseudo-polynomial time algorithm for PSP-2-DAG with $\boldsymbol{\mu} \in \mathbb{Z}_+^m$ and $b \in \mathbb{Q}_+^m$.

**Proposition 3.** *An optimal solution of PSP-2-DAG with* $b \in \mathbb{Q}_+$, $\boldsymbol{\mu} \in \mathbb{Z}_+^m$, *and* $\boldsymbol{\sigma^2} \in \mathbb{Q}_+^m$ *can be obtained in* $O(mn \sum_{a \in A} \mu_a)$ *pseudo-polynomial time.*

*Proof.* As in the proof of Proposition 2, consider the following subproblem for each $k \in \{\lfloor b \rfloor + 1, \ldots, U\}$ where $U = \sum_{a \in A} \mu_a$, similar to (1), defined as

$$z_k = \max\left\{\sum_{a \in A} \sigma_a^2 x_a : \sum_{a \in A} \mu_a x_a \leq k, x \in X_{st}\right\}, \tag{2}$$

where $z_k = -\infty$ if the subproblem is infeasible. In contrast to PSP-1, $\sum_{a \in A} \sigma_a^2$ should be maximized because $\sum_{a \in A} \mu_a x_a > b$ for all $x \in X_{st}$. Now, let $\hat{z} = \max\{(b-k)/\sqrt{z_k} : k = \lfloor b \rfloor + 1, \ldots, U \text{ such that } z_k > -\infty\}$ and $\hat{x}$ be the corresponding solution. We claim that $\Phi(\hat{z})$ and $\hat{x}$ are the optimal value of PSP-2-DAG and the corresponding optimal solution, respectively. Assume that $x^*$ is an optimal solution of PSP-2-DAG, but there is no $k$ such that $x^*$ is optimal to (2). It means $\sum_{a \in A} \sigma_a^2 x_a^* < z_{\hat{k}}$, where $\hat{k} = \sum_{a \in A} \mu_a x_a^*$. Then it contradicts the optimality of $x^*$ since $(b-\hat{k})/\sqrt{z_{\hat{k}}} > (b-\hat{k})/\sqrt{\sum_{a \in A} \sigma_a^2 x_a^*}$.

Since maximizing $\sum_{a\in A}\sigma_a^2$ is equivalent to minimizing $-\sum_{a\in A}\sigma_a^2$, the subproblem (2) for each $k \in \{\lfloor b\rfloor+1,\ldots,U\}$ is a special case of the resource constrained shortest path problem (Beasley & Christofides, 1989). In this case, all arc lengths are nonpositive because $-\sigma_a^2 \le 0$ for all $a \in A$, so that the obtained solution by the DP algorithm of Beasley & Christofides (1989) is not guaranteed to be a simple path in the presence of negative cycles. However, there is no negative cycle for PSP-2-DAG because the underlying graph is assumed to be directed acyclic. Therefore, as in the proof of Proposition 2, we can get $z_k$ and the corresponding solution for all $k \in \{\lfloor b\rfloor+1,\ldots,U\}$ by applying the DP algorithm to the subproblem (2) for $k = U$, which can be done in $O(mnU)$. Then, to find $\hat{k}$ such that $(b-\hat{k})/\sqrt{z_{\hat{k}}}$ is the maximum, we compare instead $(b-k)^2/z_k$ values to avoid the calculations of square roots, which can be done in $O(U)$. Note that in this case we have to find $\hat{k}$ such that $(b-\hat{k})^2/z_{\hat{k}}$ is the minimum because $b-k<0$ for all $k = \lfloor b\rfloor+1,\ldots,U$. Therefore, an optimal solution to PSP-2-DAG with $\boldsymbol{\mu} \in \mathbb{Z}_+^m$ can be obtained in $O(mnU)$. □

Now, we show that there exists a nontrivial special case of PSP-1 for which an optimal solution can be obtained in polynomial time. Suppose that each arc can be classified into one of $p$ classes according to its variance $\sigma_a^2$, where $p$ is a fixed positive integer, that is, $\sigma_a^2 \in \{\bar{\sigma}_1^2,\bar{\sigma}_2^2,\ldots,\bar{\sigma}_p^2\}$ for each $a \in A$. For this special case of PSP-1, which we call PSP-1($p$), $A$ can be partitioned into a fixed number of mutually disjoint subsets $A_j \subseteq A$, where $A_j = \{a \in A : \sigma_a^2 = \bar{\sigma}_j^2\}$ for all $j \in \{1,\ldots,p\}$. Then, the optimal objective value of PSP-1($p$) is $\Phi(z^*)$, where

$$z^* = \max \frac{b - \sum\limits_{a\in A}\mu_a x_a}{\sqrt{\sum_{j=1}^{p}\bar{\sigma}_j^2\cdot\left(\sum\limits_{a\in A_j} x_a\right)}} \quad \text{s.t.} \quad x \in X_{st}.$$

**Proposition 4.** *For a fixed positive integer p, an optimal solution of PSP-1(p) with $b \in \mathbb{Q}_+$, $\boldsymbol{\mu} \in \mathbb{Q}_+^m$, and $\boldsymbol{\sigma^2} \in \mathbb{Q}_+^m$ can be obtained in $O(mn^{p+1})$.*

*Proof.* For a given nonnegative integer vector $\mathbf{k} = (k_1,\ldots,k_p)$ such that $k_j \le n-1$ for all $j = 1,\ldots,p$, consider the following subproblem

$$z_{\mathbf{k}} = \min\left\{\sum_{a\in A}\mu_a x_a : \sum_{a\in A_j} x_a \le k_j,\, j=1,\ldots,p,\, x \in X_{st}\right\}, \tag{3}$$

where $z_{\mathbf{k}} = \infty$ if the subproblem is infeasible. Let $Q = \{(k_1,\ldots,k_p) \in \mathbb{Z}_+^p : \sum_{j=1}^p k_j \le n-1\}$, and let $\hat{z} = \max\{(b-z_{\mathbf{k}})/\sqrt{\sum_{j=1}^p \bar{\sigma}_j^2 k_j} : \mathbf{k} \in Q \text{ such that } z_{\mathbf{k}} < \infty\}$. Then, it can be readily seen that the optimal objective value of PSP-1($p$) is equal to $\Phi(\hat{z})$. The above subproblem (3) is a special case of the resource constrained shortest path problem, which can be solved in $O(mnD)$ by the DP algorithm proposed by Beasley & Christofides (1989) because $\mu_a \geq 0$ for all $a \in A$, where $D = k_1 \cdots k_p$ is the number of possible labels. For a given $(k_1, \ldots, k_p) \in \mathbb{Z}_+^p$, the DP algorithm of Beasley & Christofides (1989) actually solves the subproblems (3) for all $(\bar{k}_1, \ldots, \bar{k}_p)$ such that $\bar{k}_j \leq k_j$ for all $j = 1, \ldots, p$. It means that, by setting $k_j = n - 1$ for all $j = 1, \ldots, p$ and applying the DP algorithm to the corresponding subproblem (3), we can get $z_{\mathbf{k}}$ and the corresponding solution for all $\mathbf{k} \in Q$. Then, to find $\hat{\mathbf{k}}$ such that $(b - z_{\hat{\mathbf{k}}}) / \sqrt{\sum_{j=1}^{p} \bar{\sigma}_j^2 \hat{k}_j}$ is the maximum, we compare instead $(b - z_{\mathbf{k}})^2 / \sum_{j=1}^{p} \bar{\sigma}_j^2 k_j$ values to avoid the calculations of square roots, which can be done in $O(D)$. Therefore, PSP-1($p$) can be solved in $O(mn^{p+1})$ since the number of possible labels is $O(n^p)$. □

## 3. FPTAS for PSP-1

In this section, we present a fully polynomial time approximation scheme for PSP-1 with $b \in \mathbb{Q}_+$, $\boldsymbol{\mu} \in \mathbb{Q}_+^m$, and $\boldsymbol{\sigma^2} \in \mathbb{Q}_+^m$, which is defined on a directed graph (not necessarily acyclic). We assume that $\boldsymbol{\sigma} := (\sigma_{a(1)}, \ldots, \sigma_{a(m)})$ is given as a part of the input data and $\boldsymbol{\sigma} \in \mathbb{Q}_+^m$. Recall that, for PSP-1, there exists a path $P \in \mathcal{P}$ such that $\mu(P) \leq b$, which means $\rho^* \geq 0.5$, where $\rho^* = \max_{P \in \mathcal{P}} Pr\,(l(P) \leq b)$. Note also that, for a given instance of PSP-1, checking whether $\rho^* = 1$ or not can be done in polynomial time: For the given instance of PSP-1, consider a deterministic shortest path problem defined on the modified graph by deleting arcs with positive variances, which can be solved in polynomial time because $\mu_a \geq 0$ for all $a \in A$. It is clear that $\rho^* = 1$ if and only if the length of a shortest $s$-$t$ path obtained by solving the deterministic shortest path problem is less than or equal to $b$. Hence, we assume that $\rho^* \in [0.5, 1)$ without loss of generality.

The basic idea of the proposed FPTAS for PSP-1 is to iteratively solve the associated feasibility problem defined as follows. For a given $0.5 \leq \rho < 1$, let $F(\rho) = \{x \in X_{st} : Pr(\sum_{a \in A} l_a x_a \leq b) \geq \rho\}$, which is equivalent to $\{x \in X_{st} : \sum_{a \in A} \mu_a x_a + \Phi^{-1}(\rho) \sqrt{\sum_{a \in A} \sigma_a^2 x_a} \leq b\}$, where $\Phi^{-1}(\cdot)$ is the inverse of the standard normal (cumulative) distribution function.

**Definition 1.** *For a given instance of PSP-1 and a constant* $\rho \in [0.5, 1)$*, PSP-D is to determine whether or not* $F(\rho) \neq \emptyset$*, and to find* $x \in F(\rho)$ *if* $F(\rho) \neq \emptyset$*.*

Note that $F(\rho_2) \subseteq F(\rho_1)$ if $\rho_1 \leq \rho_2$ because $\Phi^{-1}(\cdot)$ is monotone increasing. Now, suppose that we know a lower bound $lb$ and an upper bound $ub$ of $\rho^*$ for a given instance of PSP-1 such that $0.5 \leq lb < ub < 1$. Then, according to the answer to PSP-D with $\rho = (lb + ub)/2$, either $\rho^* \in [lb, \rho)$ or $\rho^* \in [\rho, ub)$; either way the length of the resulting interval is half that of $[lb, ub)$. Since $\rho^* \in [0.5, 1)$ for any instance of PSP-1, starting with $[0.5, 1)$ and repeatedly bisecting the interval reduces the length of interval to $\epsilon$ in $O(\log(1/\epsilon))$ iterations for any given $\epsilon > 0$. By taking advantage of the fact that $\rho^* \geq 0.5$, it can be shown that a feasible solution with its objective value $\hat{\rho}$ such that $\rho^* - \hat{\rho} \leq \epsilon$ is an approximate solution to PSP-1 with a relative error $2\epsilon$, i.e., $\hat{\rho} \geq (1 - 2\epsilon)\rho^*$. Therefore, if we have a polynomial time algorithm for PSP-D, then we have an FPTAS for PSP-1 by calling the polynomial time algorithm for PSP-D with $O(\log(1/\epsilon))$ different values of $\rho$. The issue is that we do not know whether PSP-D is polynomially solvable. However, for any given $\alpha > 0$, we can devise an *additive* $|\alpha|$*-approximation algorithm* for PSP-D whose complexity is bounded by a polynomial function of $n$, $m$, and $1/\alpha$, which is presented in Section 3.2.

**Definition 2.** *For a given instance of PSP-D and a positive number $\alpha \leq \rho$, an additive $|\alpha|$-approximate solution to the instance is $x \in X_{st}$ such that $Pr(\sum_{a\in A} l_a x_a \leq b) \geq \rho - \alpha$. An algorithm for PSP-D is an additive $|\alpha|$-approximation algorithm, if for all instances of PSP-D, it either gives an additive $|\alpha|$-approximate solution, or guarantees that $\{x \in X_{st} : Pr(\sum_{a\in A} l_a x_a \leq b) \geq \rho\} = \emptyset$.*

**Proposition 5.** *For any given $\epsilon > 0$, an approximate solution $\hat{x} \in X_{st}$ to an instance of PSP-1 with $\hat{\rho} = Pr(\sum_{a\in A} l_a \hat{x}_a \leq b)$ such that $\hat{\rho} \geq (1 - 2\epsilon)\rho^*$ can be obtained by applying an additive $|\epsilon/2|$-approximation algorithm to $O(\log(1/\epsilon))$ instances of PSP-D.*

*Proof.* Starting with a feasible solution $x^0 \in X_{st}$ with an initial interval $[lb_0, ub_0)$, we iteratively generate a sequence of feasible solutions $x^i \in X_{st}$ along with updated intervals $[lb_i, ub_i)$ such that $\rho^* \in [lb_i, ub_i)$ and $\hat{\rho}_i \in [lb_i, ub_i)$ for $i \in \{0, 1, 2, \ldots\}$ until $ub_T - lb_T \leq \epsilon$ for some $T \geq 0$, where $\hat{\rho}_i = Pr(\sum_{a\in A} l_a x^i_a \leq b)$ for $i \in \{0, 1, 2, \ldots\}$. Recall that by calling a deterministic shortest path problem we can find $x^0 \in X_{st}$ such that $\hat{\rho}_0 \geq 0.5$ for any instance of PSP-1 in polynomial time, and that $\rho^* \in [0.5, 1)$. Hence, it is sufficient to set $lb_0 = 0.5$ and $ub_0 = 1$.

Now, suppose that, after iteration $i \geq 0$, we have $x^i \in X_{st}$ such that $\hat{\rho}_i \in [lb_i, ub_i)$ and $\rho^* \in [lb_i, ub_i)$. Then, at the next iteration $i + 1$, we apply the additive $|\epsilon/2|$-approximation algorithm to PSP-D with $\rho = (ub_i + lb_i + \epsilon/2)/2$ which is the midpoint of an adjusted interval $[lb_i + \epsilon/2, ub_i)$. As a result, we either get an additive $|\epsilon/2|$-approximate solution $x^{i+1} \in X_{st}$ such that $\hat{\rho}_{i+1} \geq \rho - \epsilon/2$ or prove that $F(\rho) = \emptyset$. In the former case, we set $lb_{i+1} = \rho - \epsilon/2$ and $ub_{i+1} = ub_i$, and in the latter we set $x^{i+1} = x^i$, $lb_{i+1} = lb_i$, and $ub_{i+1} = \rho$ (see Figure 2). In either case, we have $(ub_i - lb_i - \epsilon/2) = 2(ub_{i+1} - lb_{i+1} - \epsilon/2)$ and $\rho^* - \hat{\rho}_{i+1} \leq ub_{i+1} - lb_{i+1}$.

Therefore, if we repeat the iteration $T$ times until $\rho^* - \hat{\rho}_T \leq ub_T - lb_T \leq \epsilon$, then $\hat{\rho}_T \geq \rho^* - \epsilon \geq \rho^*(1 - 2\epsilon)$ because $\rho^* - \epsilon = \rho^*(1 - \epsilon/\rho^*)$ and $\rho^* \geq 1/2$. Since $(ub_0 - lb_0 - \epsilon/2) < 0.5$, we can see that the number of

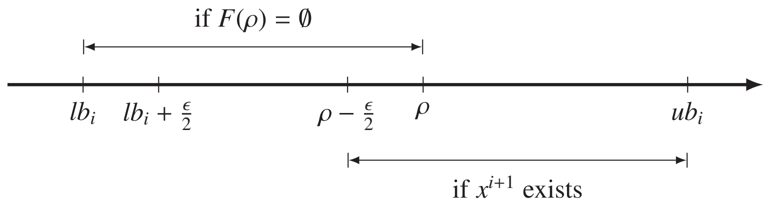



Figure 2: Interval update at iteration $i$

required iterations ($T$) does not have to be greater than $\lceil \log(1/\epsilon) \rceil$ in order to make $ub_T - lb_T - \epsilon/2 \leq \epsilon/2$, i.e., $ub_T - lb_T \leq \epsilon$. □

Proposition 5 means that an FPTAS for PSP-1 can be devised once we have a family of additive $|\alpha|$-approximation algorithms for PSP-D over all $0 < \alpha < 1$ whose time complexity is bounded by a polynomial function of $n$, $m$, and $1/\alpha$. We present such an additive $|\alpha|$-approximation algorithm for PSP-D, which is based on a polyhedral approximation of $F(\rho)$ for $\rho \in [0.5, 1)$. In the remainder of this section, we first present a polyhedral approximation of $F(\rho)$ in Section 3.1. Based on the results, an additive $|\alpha|$-approximation algorithm for PSP-D whose time complexity is bounded by a polynomial function of $n$, $m$, and $1/\alpha$ is devised in Section 3.2. Finally, an FPTAS for PSP-1 is presented in Section 3.3.

### *3.1. Polyhedral approximation of $F(\rho)$*

For a given instance of PSP-D with $\rho \in [0.5, 1)$, $F(\rho)$ is equivalent to the following set (Ben-Tal & Nemirovski, 1999)

$$\left\{ x \in X_{st} : \sum_{a \in A} \zeta_a x_a \leq b, \forall (\zeta_{a(1)}, \ldots, \zeta_{a(m)}) \in \mathcal{U}(\rho) \right\},$$

where

$$\mathcal{U}(\rho) = \left\{ \boldsymbol{\mu} + \Sigma^{1/2} y : \sum_{a \in A} y_a^2 \leq (\Phi^{-1}(\rho))^2, y = (y_{a(1)}, \ldots, y_{a(m)}) \in \mathbb{R}_+^m \right\} \text{ and } \Sigma = \begin{bmatrix} \sigma_{a(1)}^2 & & \\ & \ddots & \\ & & \sigma_{a(m)}^2 \end{bmatrix}.$$

By using a piecewise linear approximation $L_r(y_a)$ of the quadratic loss function $f(y_a) = y_a^2$ for each $a \in A$ for a given positive integer $r$, as shown in Figure 3, $\mathcal{U}(\rho)$ can be approximated as a polyhedron (Han et al., 2016). Suppose that the interval $[0, (\Phi^{-1}(\rho))^2]$ along the vertical axis is divided into $r$ segments, $[0, h_a^1], [h_a^1, h_a^1 + h_a^2], \ldots, [\sum_{k=1}^{r-1} h_a^k, \sum_{k=1}^{r} h_a^k]$, and let the corresponding lengths of divided horizontal

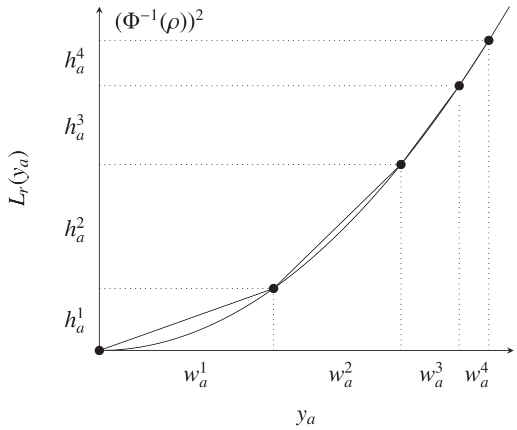



Figure 3: Piecewise linear approximation for $r = 4$

intervals be $w_a^1, w_a^2, \ldots, w_a^r$. As can be seen in Figure 3, $h_a^k$ is the length of the $k$th divided segment of the vertical interval $[0, (\Phi^{-1}(\rho))^2]$, and the length of the corresponding divided horizontal interval, $w_a^k$, is equal to $\sqrt{\sum_{i=1}^{k} h_a^i} - \sqrt{\sum_{i=1}^{k-1} h_a^i}$. Then, a piecewise linear approximation $L_r(y_a)$ of $f(y_a) = y_a^2$ over $[0, \Phi^{-1}(\rho)]$ can be defined as

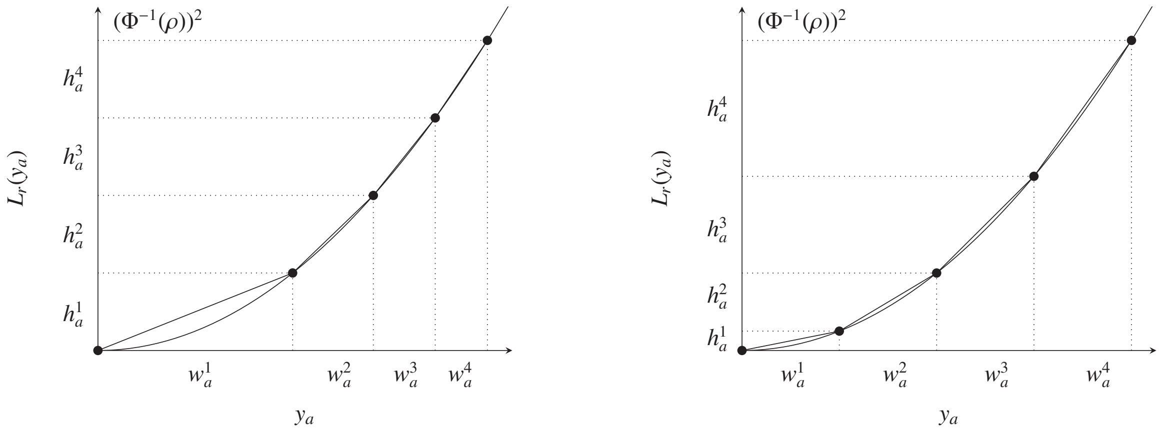
$$L_r(y_a) = \frac{h_a^k}{w_a^k}\left(y_a - \sum_{i=1}^{k} w_a^i\right) + \sum_{i=1}^{k} h_a^i, \tag{4}$$

for $\sum_{i=1}^{k-1} w_a^i \leq y_a \leq \sum_{i=1}^{k} w_a^i$ and $k \in R$, where $\sum_{k=1}^{r} w_a^k = \Phi^{-1}(\rho)$ and $R = \{1, \ldots, r\}$ is the set of linear segments. For a given positive integer $r$, depending on how the interval $[0, (\Phi^{-1}(\rho))^2]$ along the vertical axis (or equivalently the interval $[0, \Phi^{-1}(\rho)]$ along the horizontal axis) is divided, infinitely many piecewise linear approximations are possible. Han et al. (2016) used a piecewise linear approximation for the quadratic loss function with $h_a^1 = h_a^2 = \cdots = h_a^r$ for the chance-constrained knapsack problem, and Ryu & Park (2021) mentioned one with $w_a^1 = w_a^2 = \cdots = w_a^r$ for the distributionally robust chance-constrained knapsack problem. Figure 4 shows these two examples of the piecewise linear approximation defined as (4).

For a given piecewise linear approximation (4) with $r$ linear segments, let us define the maximum additive error of the approximation as

$$\max\left\{L_r(y_a) - y_a^2 : 0 \leq y_a \leq \Phi^{-1}(\rho)\right\}. \tag{5}$$

For our purpose, we use the piecewise linear approximation obtained by equally dividing the interval $[0, \Phi^{-1}(\rho)]$ along the horizontal axis, because it is optimal in terms of the maximum error (5) as stated in the following proposition. The proof of Proposition 6 is given in the Appendix.




(a) $h_a^1 = h_a^2 = \cdots = h_a^r = (\Phi^{-1}(\rho))^2/r$

(b) $w_a^1 = w_a^2 = \cdots = w_a^r = (\Phi^{-1}(\rho))/r$

Figure 4: Examples of the piecewise linear approximation for $r = 4$

**Proposition 6.** *For a given number of linear segments r, the maximum error (5) of a piecewise linear approximation defined as (4) is minimized when* $w_a^1 = w_a^2 = \ldots = w_a^r = \Phi^{-1}(\rho)/r$*, and the minimum error is* $(\Phi^{-1}(\rho))^2/4r^2$.

Now, for each $a \in A$, let $L_r^*(y_a)$ be the piecewise linear approximation (4) such that $w_a^1 = w_a^2 = \cdots = w_a^r = \Phi^{-1}(\rho)/r$. Then, we define $\mathcal{U}_r(\rho)$ and $F_r(\rho)$ as

$$\mathcal{U}_r(\rho) = \left\{\mu + \Sigma^{1/2}y : \textstyle\sum_{a\in A} L_r^*(y_a) \le (\Phi^{-1}(\rho))^2, y = (y_{a(1)}, \ldots, y_{a(m)}) \in \mathbb{R}_+^m\right\}, \tag{6}$$

$$F_r(\rho) = \left\{x \in X_{st} : \textstyle\sum_{a\in A} \zeta_a x_a \le b, \forall(\zeta_{a(1)}, \ldots, \zeta_{a(m)}) \in \mathcal{U}_r(\rho)\right\}. \tag{7}$$

Since, for each $a \in A$, $L_r^*(y_a) \ge f(y_a)$ for all $y_a \in [0, \Phi^{-1}(\rho)]$, it is clear that $\mathcal{U}_r(\rho) \subseteq \mathcal{U}(\rho)$ and $F(\rho) \subseteq F_r(\rho)$. It means that $F_r(\rho) = \emptyset$ implies $F(\rho) = \emptyset$. Moreover, if there exists $\bar{x} \in F_r(\rho)$, then $\rho - Pr(\sum_{a\in A} l_a \bar{x}_a \le b) \le \rho - \rho_r$, where $\rho_r = \inf\{Pr(\sum_{a\in A} l_a x_a \le b) : x \in F_r(\rho)\}$. Therefore, if we can guarantee $\rho - \rho_r \le \alpha$ for any given $\alpha \in (0,1)$, we get an additive $|\alpha|$-approximate solution to PSP-D by checking whether or not $F_r(\rho) \neq \emptyset$.

For any given number of linear segments $r$, we show that checking whether or not $F_r(\rho) \neq \emptyset$ can be done efficiently in Section 3.2. The issue is how many linear segments are needed to make sure that $\rho - \rho_r \le \alpha$ for any given $\alpha \in (0,1)$. The following Proposition 7, whose proof is given in the Appendix, gives an upper bound on $\rho - \rho_r$ which approaches zero as $r$ increases.

**Proposition 7.** *For any given instance of PSP-D, if* $F_r(\rho) \neq \emptyset$ *with the number of linear segments* $r > \sqrt{m/4}$*, then* $\rho - \rho_r \le \frac{m}{\sqrt{2\pi e}(4r^2 - m)}$.

From Proposition 7, we can show that the minimum number of required linear segments $r$ to guarantee $\rho - \rho_r \leq \alpha$ for any $\alpha \in (0, 1)$ is $\Theta(\sqrt{m/\alpha})$, i.e., $\Omega(\sqrt{m/\alpha})$ as well as $O(\sqrt{m/\alpha})$.

**Corollary 1.** *For any given instance of PSP-D and* $\alpha \in (0, 1)$, *if suffices to set*

$$r = \left\lceil \sqrt{m/4 + m/(4\alpha\sqrt{2\pi e})} \right\rceil$$

*to guarantee that* $\rho - \rho_r \leq \alpha$, *which is independent of the value of* $\rho$.

*Proof.* By Proposition 7, it suffices to set $r$ to the smallest integer such that $r^2 \geq m/4 + Cm/\alpha$, where $C = 1/(4\sqrt{2\pi e})$. Note that $C < 1/16$. Therefore, the minimum $r$ required is $\Theta(\sqrt{m/\alpha})$. □

Therefore, by Corollary 1, we can get an additive $|\alpha|$-approximate solution to PSP-D for any given $\alpha \in (0, 1)$ by checking whether or not $F_r(\rho) \neq \emptyset$ with $r$ being set as in Corollary 1. Note that the minimum number of linear segments to guarantee $\rho - \rho_r \leq \alpha$ is independent of $\rho$, which is sufficient to devise an FPTAS for PSP-1 given in Section 3.3. Han et al. (2016) also gave similar results for their piecewise linear approximation. However, they showed the minimum number of linear segments to guarantee $\rho - \rho_r \leq \alpha$ is proportional to $\Phi^{-1}(\rho)m/\alpha$, which goes to the infinity as $\rho$ gets closer to 1. Therefore, our results given in Proposition 7 and Corollary 1 are not direct consequences of the results of Han et al. (2016). Moreover, Corollary 1 significantly reduces the number of linear segments required to guarantee $\rho - \rho_r \leq \alpha$ for any given $\alpha \in (0, 1)$.

### *3.2. Additive $|\alpha|$-approximation algorithm for PSP-D*

As mentioned earlier, we claim that for a given number of linear segments $r$, checking whether or not $F_r(\rho) \neq \emptyset$ can be done efficiently. To show that, we first give another representation of $F_r(\rho)$ defined as (7) in the following proposition. The proof of the proposition is given in the Appendix.

**Proposition 8.**

$$F_r(\rho) = \left\{ x \in X_{st} : \sum_{a \in A} \mu_a x_a + \beta(\rho, x, r) \leq b \right\}, \tag{8}$$

*where*

$$\beta(\rho, x, r) = \max_{z \in \mathbb{R}^{mr}} \quad \sum_{a \in A} \sum_{k \in R} d_a^k x_a z_a^k \tag{9}$$

$$s.t. \quad \sum_{a \in A} \sum_{k \in R} h_a^k z_a^k \leq (\Phi^{-1}(\rho))^2, \tag{10}$$

$$z_a^k \leq \lfloor z_a^{k-1} \rfloor, \quad \forall a \in A, k \in R \setminus \{1\}, \tag{11}$$

$$0 \leq z_a^k \leq 1, \quad \forall a \in A, k \in R, \tag{12}$$

*and* $d_a^k = \sigma_a w_a^k$ *for all* $a \in A, k \in R$.

Now, Proposition 9 whose proof is given in the Appendix shows that checking whether or not $F_r(\rho) \neq \emptyset$ can be done by solving $mr + 1$ deterministic shortest path problems with nonnegative arc lengths.

Let $D = \{(i,k) : i \in I_A, k \in R\} \cup \{(m+1, 1)\}$, where $(m+1, 1)$ is an artificial segment with $d^1_{a(m+1)} = 0$ and $h^1_{a(m+1)}$ an arbitrary positive number. Then, we make a sorted list of the set $\{d^k_{a(i)}/h^k_{a(i)} : (i,k) \in D\}$, denoted as $\{s_1, s_2, \dots, s_{mr+1}\}$, in the descending order where $d^k_{a(p)}/h^k_{a(p)}$ comes before $d^l_{a(i)}/h^l_{a(i)}$ if $p < i$ for any pair of $(p,k)$ and $(i,l)$ with $d^k_{a(p)}/h^k_{a(p)} = d^l_{a(i)}/h^l_{a(i)}$. Let $Q = \{1, \dots, mr+1\}$ be the index set of the sorted list. For each $q \in Q$, consider the following problem $\text{SP}_q$, which is an deterministic shortest path problem defined on the same graph as the given instance of PSP-D but with possibly different nonnegative arc lengths.

$$(\text{SP}_q) \quad \min \quad \sum_{a \in A} \mu_a x_a + \sum_{(i,k) \in D_q} (d^k_{a(i)} - h^k_{a(i)} s_q) x_{a(i)}$$
$$\text{s.t.} \quad x \in X_{st},$$

where $D_q = \{(i,k) \in D : d^k_{a(i)}/h^k_{a(i)} > s_q\}$. Note that $(d^k_{a(i)} - h^k_{a(i)} s_q) > 0$ for all $(i,k) \in D_q$.

**Proposition 9.** *For a given instance of PSP-D and a positive integer r,* $F_r(\rho) \neq \emptyset$ *if and only if there exists* $q \in Q = \{1, \dots, mr+1\}$ *such that the optimal objective value of* $SP_q$ *is less than or equal to* $b - (\Phi^{-1}(\rho))^2 s_q$.

Based on the previous results, we can devise a family of additive $|\alpha|$-approximation algorithms for PSP-D over all $0 < \alpha < 1$ whose time complexity is bounded by a polynomial function in $n$, $m$ and $1/\alpha$. Note that, for given $\rho \in [0.5, 1)$, $\Phi^{-1}(\rho)$ may not be a rational number. Hence, we assume that, for any given $\alpha > 0$, a rational number $\phi \geq 0$ such that $0 \leq \Phi^{-1}(\rho) - \phi \leq \alpha$ can be obtained in $O(1)$. Note that $\phi = \Phi^{-1}(\tilde{\rho})$ for some $\tilde{\rho} \in [0.5, \rho]$. Since the probability density function of the standard normal random variable, $f(u)$, is strictly decreasing for $u \geq 0$ and $f(0) < 1/2$, it is clear that $0 \leq \rho - \tilde{\rho} \leq \alpha/2$.

**Theorem 1.** *For a given instance of PSP-D and* $\alpha \in (0,1)$*, an additive* $|\alpha|$*-approximate solution can be found in* $O((m + n \log n) \cdot m^{1.5} \sqrt{1/\alpha})$.

*Proof.* Recall that $\rho \in [0.5, 1)$. First, get a rational number $\phi \geq 0$ such that $0 \leq \Phi^{-1}(\rho) - \phi \leq \alpha$. Let $\tilde{\rho} \in [0.5, \rho]$ be a real number such that $\phi = \Phi^{-1}(\tilde{\rho})$. Since $\tilde{\rho} \leq \rho$, $F_r(\rho) \subseteq F_r(\tilde{\rho})$. In the subsequent computations, only $\phi$ is needed, so that we do not have to explicitly compute $\tilde{\rho}$. Next, we compute the number of linear segments $r$ needed to guarantee $\tilde{\rho} - \rho_r \leq \alpha/2$. According to Corollary 1 with parameter set to $\alpha/2$,

$$r \geq \sqrt{m/4 + m/(2\alpha\sqrt{2\pi e})}.$$

Since $2\sqrt{2\pi e} > 8$, it is clear that $\sqrt{m/4 + m/(2\alpha\sqrt{2\pi e})} < \sqrt{m/4 + m/(8\alpha)}$. Hence, it is sufficient to set $r$ to the smallest integer such that $r^2 \geq m/4 + m/8\alpha$. Now, to determine whether $F_r(\tilde{\rho}) = \emptyset$ or not, construct and solve the corresponding $\text{SP}_q$ for each $q \in Q = \{1, \ldots, mr+1\}$. If the optimal objective value of $\text{SP}_q$ is greater than $b - \phi^2 s_q$ for all $q \in Q$, then $F_r(\tilde{\rho}) = \emptyset$ by Proposition 9, which means $F(\rho) = \emptyset$ because $F(\rho) \subseteq F_r(\rho)$ and $F_r(\rho) \subseteq F_r(\tilde{\rho})$. Otherwise, we get $\hat{x} \in F_r(\tilde{\rho})$ that satisfies $\tilde{\rho} - \hat{\rho} \leq \alpha/2$, where $\hat{\rho} = Pr(\sum_{a\in A} l_a \hat{x}_a \leq b)$. Since $\rho - \tilde{\rho} \leq \alpha/2$ as explained right before the statement of Theorem 1, $\rho - \hat{\rho} \leq \alpha$, which means $\hat{x}$ is an additive $|\alpha|$-approximate solution to the given instance of PSP-D. Note that the minimum number of required segments $r = \Theta(\sqrt{m/\alpha})$ by Corollary 1, and that $\text{SP}_q$ for each $q \in Q$ is a deterministic shortest path problem with nonnegative arc lengths which can be solved in $O(m + n\log n)$ by a Fibonacci heap implementation of Dijkstra's algorithm (Fredman & Tarjan, 1987). Therefore, the total time complexity is $O((m + n\log n) \cdot m^{1.5}\sqrt{1/\alpha})$. □

### *3.3. FPTAS for PSP-1*

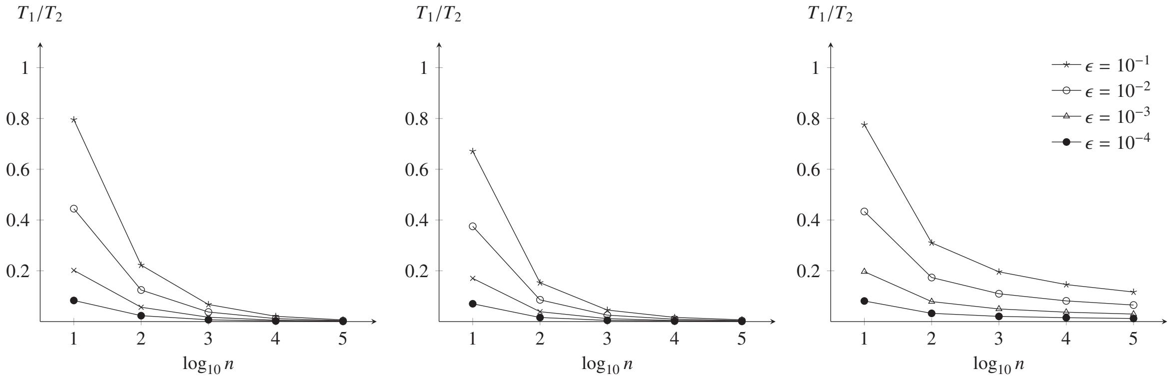
Now, we can devise an FPTAS for PSP-1 based on Proposition 5 and Theorem 1.

**Theorem 2.** *For any given $\epsilon > 0$, an approximate solution $\hat{x} \in X_{st}$ to PSP-1 with $\hat{\rho} = Pr(\sum_{a\in A} l_a \hat{x}_a \leq b)$ such that $\hat{\rho} \geq (1 - 2\epsilon)\rho^*$ can be obtained in $O((m + n\log n) \cdot m^{1.5}\sqrt{1/\epsilon}\log(1/\epsilon))$.*

*Proof.* Apply the bisection procedure given in the proof of Proposition 5 with $lb_0 = 0.5$ and $ub_0 = 1$. At each iteration, apply the additive $|\epsilon/2|$-approximation algorithm given in the proof of Theorem 1 to the corresponding instance of PSP-D. Then, by Proposition 5, we can get $\hat{x} \in X_{st}$ such that $\hat{\rho} \geq (1 - 2\epsilon)\rho^*$, where $\hat{\rho} = Pr(\sum_{a\in A} l_a \hat{x}_a \leq b)$ in $O(\log(1/\epsilon))$ iterations, and the time complexity of each iteration is $O((m + n\log n) \cdot m^{1.5}\sqrt{1/\epsilon})$ by Theorem 1. Therefore, the result follows. □

The time complexity of the FPTAS for PSP-1 given in Theorem 2 is strongly polynomial, that is, it is independent of the magnitude of numeric values of the given data such as means and variances of arc lengths. Furthermore, the time complexity is sublinear in $1/\epsilon$ for $\epsilon \in (0, 1)$, since $\sqrt{1/\epsilon}\log(1/\epsilon) = O((1/\epsilon)^{0.5+\beta})$ for any $\beta \in (0, 0.5)$.

Let $T_1$ and $T_2$ be time complexities of our FPTAS and that of Xiao et al. (2011) to obtain an approximate solution such that $\hat{\rho} \geq (1 - \epsilon)\rho^*$, respectively. Then $T_1 \approx (m + n\log_2 n) \cdot m^{1.5}\sqrt{2/\epsilon}\log_2(2/\epsilon)$ by Theorem 2 and $T_2 \approx (1/\epsilon) \cdot m^2 n\log_2 n$. Figure 5 shows $T_1/T_2$ for $\epsilon = 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}$ and $n = 10^1, 10^2, \ldots, 10^5$ on sparse ($m = n$), moderate ($m = n^{1.5}$) and dense ($m = n^2$) graphs. As can be seen, our scheme is faster than Xiao et al. (2011), and it gets better as $n$ increases and $\epsilon$ decreases.




(a) sparse ($m = n$) (b) moderate ($m = n^{1.5}$) (c) dense ($m = n^2$)

Figure 5: Ratio of the time complexity of our scheme ($T_1$) and Xiao et al. (2011) ($T_2$)

Even though the approximation scheme given in Theorem 2 is devised for PSP-1, its structure, which is based on the bisection procedure given in Proposition 5 together with the approximation algorithm for the corresponding feasibility problem, is generic in that it can be applied to other combinatorial optimization problems.

Suppose that we have a set of items $A = \{1, \ldots, m\}$ with $l_i \in \mathbb{Q}_+$ for each $i \in A$. Consider a combinatorial optimization problem (CO) to find $x \in X$ that minimizes $\sum_{i \in A} l_i x_i$, where $X \subseteq \mathbb{B}^m$. When objective coefficients $l_i$ for $i \in A$ are uncertain and they can be modelled as independent normal random variables, the corresponding probability maximization problem (PCO) can be defined as the problem of maximizing $Pr(\sum_{i \in A} l_i x_i \leq b)$ over $x \in X$, where $b \in \mathbb{Q}_+$ and $l_i$ is an independent normal random variable with $\mu_i \in \mathbb{Q}_+$ and $\sigma_i \in \mathbb{Q}_+$ for each $i \in A$. PSPP of Halman et al. (2019) also can be generalized by replacing the path set with the solution set $X$. In addition, let us define the feasibility problem, PCO-D, corresponding to PCO as the problem of determining whether or not

$$\mathcal{F}(\rho) = \left\{ x \in X : Pr\left( \sum_{i \in A} l_i x_i \leq b \right) \geq \rho \right\}$$

is empty in the same way as that we used to define PSP-D in Definition 1. Then, the following theorem shows that there exists an FPTAS for PCO, once the corresponding CO is polynomially solvable and there exists $x \in X$ such that $\sum_{i \in A} \mu_i x_i \leq b$.

**Theorem 3.** *For any instance of PCO, assume that the optimal objective value $\rho^* \geq 0.5$. Then, for any given $\epsilon > 0$, an approximate solution $\hat{x} \in X$ with $\hat{\rho} = Pr(\sum_{i \in A} l_i \hat{x}_i \leq b)$ such that $\hat{\rho} \geq (1 - 2\epsilon)\rho^*$ can be found by applying an algorithm for CO at most $O(m^{1.5} \sqrt{1/\epsilon} \log(1/\epsilon))$ times.*

*Proof.* Let $\mathcal{F}_r(\rho) = \{x \in X : \sum_{i \in A} \zeta_i x_i \leq b, \forall (\zeta_1, \ldots, \zeta_m) \in \mathcal{U}_r(\rho)\}$ and

$$(\mathrm{CO}_q) \quad \min \quad \sum_{i \in A} \mu_i x_i + \sum_{(i,k) \in D_q} (d_i^k - h_i^k s_q) x_i$$

$$\text{s.t.} \quad x \in X.$$

Then, for any given $0 < \alpha < 1$, the minimum number of required linear segments $r$ to guarantee $\rho - \rho_r \leq \alpha$ is $\Theta(\sqrt{m/\alpha})$, where $\rho_r = \inf\{Pr(\sum_{i \in A} l_i x_i \leq b) : x \in \mathcal{F}_r(\rho)\}$, which can be readily proved by replacing $X_{st}$ with $X$ in the proofs of Proposition 7 and Corollary 1. In addition, for a given instance of PCO-D and positive integer $r$, whether or not $\mathcal{F}_r(\rho) \neq \emptyset$ can be determined by solving deterministic combinatorial optimization problems $CO_q$ for $q = 1, \ldots, mr+1$. This result follows from the proofs of Propositions 8 and 9 by replacing $X_{st}$ with $X$.

Hence, for a given instance of PCO-D and $\epsilon > 0$, we can find an approximate solution $x \in X$ such that $\rho - Pr(\sum_{i \in A} l_i x_i \leq b) \leq \epsilon/2$ by solving $O(mr) = O(m^{1.5}\sqrt{1/\epsilon})$ instances of CO in the same way as described in the proof of Theorem 1. In addition, by applying the bisection procedure given in the proof of Proposition 5 to PCO, we can find an approximate solution $x \in X$ to PCO such that $Pr(\sum_{i \in A} l_i x_i \leq b) \geq (1 - 2\epsilon)\rho^*$ for any $0 < \epsilon < 1$ in $O(\log(1/\epsilon))$ iterations. Therefore, the total number of instances of CO that we need to solve is at most $O(m^{1.5}\sqrt{1/\epsilon}\log(1/\epsilon))$. □

The minimum spanning tree problem with uncertain edge lengths (Hiroaki et al., 1981) can be an example of Theorem 3. For a given undirected graph with the length of each edge being an independent normal random variable with rational mean and standard deviation, consider the problem of finding a minimum spanning tree that maximizes the probability of the total length being within a given limit. Since the deterministic minimum spanning tree problem can be solved in polynomial time, an FPTAS for the problem can be devised by Theorem 3.

## 4. Concluding Remarks

In this paper, we consider PSP with $b \in \mathbb{Q}_+$, $\boldsymbol{\mu} \in \mathbb{Q}_+^m$, and $\boldsymbol{\sigma^2} \in \mathbb{Q}_+^m$. We showed that PSP-2 with $\boldsymbol{\mu} \in \mathbb{Z}_+^m$ is $\mathcal{NP}$-hard even on directed acyclic graphs in Proposition 1. We also showed that PSP-1 and PSP-2-DAG with $\boldsymbol{\mu} \in \mathbb{Z}_+^m$ can be solved in pseudo-polynomial time in Proposition 2 and 3, respectively. In addition, we identified a nontrivial special case of PSP-1 which is polynomially solvable in Proposition 4. Even though the computational complexity of PSP-1 has not been completely settled down, an FPTAS for PSP-1 whose time complexity is strongly polynomial was proposed in Theorem 2. The structure of the proposed FPTAS is generic so that it can be applied to other combinatorial optimization problems as stated in Theorem 3.

As mentioned before, the computational complexity of PSP-1 is still open, and thus it needs to be studied further. Improving the time complexity of the proposed FPTAS could also be studied; the number of deterministic shortest path problems could be reduced by improving the total time complexity for solving $mr+1$ shortest path problem instances by exploiting similarities among those instances. In addition, general cases of PSP where arc lengths are correlated random variables could be studied.

**Acknowledgement**

The authors would like to thank the editor and anonymous reviewers for their valuable comments. The authors also would like to thank the Institute for Industrial Systems Innovation of Seoul National University for the administrative support. This work was supported by the National Research Foundation of Korea (NRF) grant funded by the Korean government (No. 2018R1A2B2003227); the National Research Foundation of Korea (NRF) grant funded by the Korean government (MSIT) (No. 2021R1A2C2005531).

**Appendix**

*Proof of Proposition 6.* Assume that the line segment from $(\bar{y}_a, \bar{y}_a^2)$ to $(\bar{y}_a + w_a^k, \bar{y}_a^2 + h_a^k)$ of Figure 6 is the $k$th linear segment of $L_r(y_a)$. Then,

$$\max\left\{L_r(y_a) - y_a^2 : \bar{y}_a \le y_a \le \bar{y}_a + w_a^k\right\} = \max\left\{L_r(y_a) - g(y_a) : \bar{y}_a \le y_a \le \bar{y}_a + w_a^k\right\},$$

where $g(y_a)$ is the affine function parallel to the $k$th linear segment and tangent to the loss function $f(y_a)$.

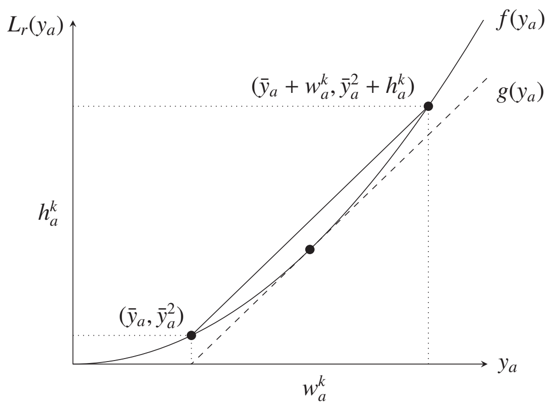



Figure 6: The $k$th linear segment of $L_r(y_a)$

By the definition (4) of $L_r(y_a)$, $h_a^k = (\bar{y}_a + w_a^k)^2 - \bar{y}_a^2 = (w_a^k)^2 + 2\bar{y}_a w_a^k$. It is clear that the slope of $g(y_a)$ is $h_a^k / w_a^k = w_a^k + 2\bar{y}_a$. By noting that the discriminant of the quadratic equation, $y_a^2 - g(y_a) = 0$, should be 0 in order for $g(y_a)$ to be tangent to the loss function $f(y_a)$, the vertical intercept of $g(y_a)$ can be obtained as $-(w_a^k + 2\bar{y}_a)^2/4$. Therefore, $g(y_a) = (w_a^k + 2\bar{y}_a)y_a - (w_a^k + 2\bar{y}_a)^2/4$. That is, the maximum error between the $k$th segment and the loss function is $\bar{y}_a^2 - g(\bar{y}_a) = (w_a^k)^2/4$, which depends only on $w_a^k$. Therefore, the maximum error (5) can be represented as

$$\max_{k \in R} \frac{(w_a^k)^2}{4},$$

and it is minimized when $w_a^1 = w_a^2 = \ldots = w_a^r = \Phi^{-1}(\rho)/r$, and the minimum is $(\Phi^{-1}(\rho))^2/4r^2$. □

*Proof of Proposition 7.* We first show that $\mathcal{V}_r(\rho) \subseteq \mathcal{U}_r(\rho)$ with $r > \sqrt{m/4}$, where $\mathcal{V}_r(\rho)$ is defined as

$$\mathcal{V}_r(\rho) = \left\{ \mu + \Sigma^{1/2} y : 0 \leq \|y\|_2 \leq \Phi^{-1}(\rho) \sqrt{1 - \frac{m}{4r^2}}, y = (y_{a(1)}, \ldots, y_{a(m)}) \in \mathbb{R}_+^m \right\}.$$

By Proposition 6, the maximum error of $L_r^*(\cdot)$ is $(\Phi^{-1}(\rho))^2/4r^2$. Hence, it is clear that for all $a \in A$

$$L_r^*(y_a) \leq y_a^2 + \frac{(\Phi^{-1}(\rho))^2}{4r^2}.$$

Therefore, If $\|y\|_2 \leq \Phi^{-1}(\rho) \sqrt{1 - m/4r^2}$,

$$\sum_{a \in A} L_r^*(y_a) \leq \sum_{a \in A} y_a^2 + \frac{m(\Phi^{-1}(\rho))^2}{4r^2} \leq (\Phi^{-1}(\rho))^2,$$

which means $\mathcal{V}_r(\rho) \subseteq \mathcal{U}_r(\rho)$. Now, since $\mathcal{V}_r(\rho) \subseteq \mathcal{U}_r(\rho)$, it is clear that $x \in F_r(\rho)$ satisfies the following constraint

$$\sum_{a \in A} \zeta_a x_a \leq b, \forall (\zeta_{a(1)}, \ldots, \zeta_{a(m)}) \in \mathcal{V}_r(\rho),$$

which is equivalent to

$$Pr\left( \sum_{a \in A} l_a x_a \leq b \right) \geq \Phi(\beta_r), \text{where } \beta_r = \Phi^{-1}(\rho) \sqrt{1 - m/4r^2}.$$

Therefore, $\rho_r \geq \Phi(\beta_r)$ and

$$\begin{aligned} \alpha_r = \rho - \rho_r \leq \rho - \Phi(\beta_r) &= \frac{1}{\sqrt{2\pi}} \int_{\beta_r}^{\Phi^{-1}(\rho)} e^{-t^2/2} dt \\ &\leq \frac{1}{\sqrt{2\pi}} (\Phi^{-1}(\rho) - \beta_r) e^{-(\beta_r)^2/2} \qquad (13) \\ &= \frac{1}{\sqrt{2\pi}} \Phi^{-1}(\rho) \left( 1 - \sqrt{1 - \frac{m}{4r^2}} \right) e^{-(\Phi^{-1}(\rho))^2 \left(1 - \frac{m}{4r^2}\right)/2}. \end{aligned}$$

Note that the second inequality (13) follows from the fact that $e^{-t^2/2}$ is monotone decreasing for $t \geq 0$. Since $\ln(\cdot)$ is a monotone increasing function,

$$\begin{aligned}\ln(\alpha_r) &\leq \ln\eta + \ln\left(1 - \sqrt{1 - \frac{m}{4r^2}}\right) - \left(\Phi^{-1}(\rho)\right)^2\left(1 - \frac{m}{4r^2}\right)\Big/2 \\ &= \ln\eta + \ln\left(1 - \sqrt{1 - \frac{m}{4r^2}}\right) - \pi\eta^2\left(1 - \frac{m}{4r^2}\right),\end{aligned}$$

where $\eta = \Phi^{-1}(\rho)/\sqrt{2\pi}$. Now, let $\theta = 1 - m/4r^2 > 0$, then we have

$$\ln(\alpha_r) \leq \ln\eta + \ln\left(1 - \sqrt{\theta}\right) - \pi\theta\eta^2.$$

If we define a function $g(\eta) = \ln\eta + \ln\left(1 - \sqrt{\theta}\right) - \pi\theta\eta^2$, then it is readily verified that $g(\eta)$ is a concave function with respect to $\eta$, and it attains its maximum at $\eta = 1/\sqrt{2\pi\theta}$. Since $\eta$ is in fact a monotone increasing function of $\rho$ and $\eta \in [0, \infty)$ for all $\rho \in [0.5, 1)$, there exists $\rho \in [0.5, 1)$ such that $\eta = \Phi^{-1}(\rho)/\sqrt{2\pi} = 1/\sqrt{2\pi\theta}$. Therefore, the following relation holds.

$$\ln(\alpha_r) \leq \ln\left(\frac{1}{\sqrt{2\pi\theta}}\right) - \pi\theta\frac{1}{2\pi\theta} + \ln\left(1 - \sqrt{\theta}\right) = \ln\left(\frac{1 - \sqrt{\theta}}{\sqrt{2\pi\theta e}}\right),$$

which is equivalent to

$$\alpha_r \leq \frac{1 - \sqrt{1 - \frac{m}{4r^2}}}{\sqrt{2\pi e\left(1 - \frac{m}{4r^2}\right)}}.$$

Since

$$\frac{1 - \sqrt{1 - \frac{m}{4r^2}}}{\sqrt{2\pi e\left(1 - \frac{m}{4r^2}\right)}} \leq \frac{\left(1 - \sqrt{1 - \frac{m}{4r^2}}\right)\left(1 + \sqrt{1 - \frac{m}{4r^2}}\right)}{\sqrt{2\pi e} \cdot \sqrt{1 - \frac{m}{4r^2}}} \leq \frac{m}{\sqrt{2\pi e}(4r^2 - m)},$$

we finally have the following inequality

$$\alpha_r \leq \frac{m}{\sqrt{2\pi e}(4r^2 - m)}.$$

□

*Proof of Proposition 8.* By the definition of $F_r(\rho)$ given as (7), we have only to show that $\boldsymbol{\zeta} \in \mathcal{U}_r(\rho)$ if and only if there exists $z \in [0, 1]^{mr}$ that satisfies (10), (11), and (12) such that $\zeta_a = \mu_a + \sum_{k \in R} d_a^k z_a^k$ for all $a \in A$, where $\boldsymbol{\zeta} = (\zeta_{a(1)}, \ldots, \zeta_{a(m)}) \in \mathbb{R}^m$. Suppose that $\boldsymbol{\zeta} \in \mathcal{U}_r(\rho)$. It means there exists $y \in \mathbb{R}_+^m$ such that

$\sum_{a\in A} L_r^*(y_a) \le (\Phi^{-1}(\rho))^2$ and $\zeta_a = \mu_a + \sigma_a y_a$ for all $a \in A$. Let $k(a) = \min\{k \in R : y_a < \sum_{i=1}^{k} w_a^i\}$. If we set

$$z_a^k = \begin{cases} 1, & \text{for } k = 1, \ldots, k(a) - 1, \\ y_a - \sum_{i=1}^{k-1} w_a^i, & \text{for } k = k(a), \\ 0, & \text{for } k = k(a) + 1, \ldots, r, \end{cases}$$

then $z \in [0,1]^{mr}$ satisfies (10), (11), and (12) and $\zeta_a = \mu_a + \sum_{k\in R} d_a^k z_a^k$ for all $a \in A$. Now, suppose that we have $z \in [0,1]^{mr}$ that satisfies (10), (11), and (12). If we set $y_a = \sum_{k=1}^{r} w_a^k z_a^k$ for all $a \in A$, then, by (10) together with the definition of $L_r^*(\cdot)$, it can be readily verified that $\sum_{a\in A} L_r^*(y_a) \le (\Phi^{-1}(\rho))^2$. Hence, $(\zeta_{a(1)}, \ldots, \zeta_{a(m)}) \in \mathcal{U}_r(\rho)$, if we set $\zeta_a = \mu_a + \sigma_a y_a$ for all $a \in A$. □

*Proof of Proposition 9.* By the definition of $L_r^*(y_a)$,we can see that $h_a^1/w_a^1 \le h_a^2/w_a^2 \le \ldots \le h_a^r/w_a^r$ for all $a \in A$. For any $x \in X_{st}$, it means $(d_a^1 x_a)/h_a^1 \ge (d_a^2 x_a)/h_a^2 \ge \ldots \ge (d_a^r x_a)/h_a^r$ for all $a \in A$. Therefore, for any $x \in X_{st}$, the optimal objective value of $\beta(\rho, x, r)$ is the same as that of a linear program obtained by deleting constraints (11). Then, by replacing $\beta(\rho, x, r)$ with its dual linear program, the constraint of (8) can be reformulated as the following multiple constraints using dual variables $\nu \in \mathbb{R}_+^1$ and $\omega \in \mathbb{R}_+^{mr}$ associated with (10) and (12), respectively:

$$\sum_{a\in A} \mu_a x_a + (\Phi^{-1}(\rho))^2 \nu + \sum_{a\in A} \sum_{k\in R} \omega_a^k \le b, \tag{14}$$

$$h_a^k \nu + \omega_a^k \ge d_a^k x_a, \quad \forall a \in A, k \in R, \tag{15}$$

$$\omega_a^k \ge 0, \quad \forall a \in A, k \in R, \tag{16}$$

$$\nu \ge 0. \tag{17}$$

Therefore, for any given $x \in F_r(\rho)$, there exist $\nu \in \mathbb{R}_+^1$ and $\omega \in \mathbb{R}_+^{mr}$ that satisfy the constraints (14) and (15). In addition, in view of the constraints (14), $\omega_a^k$ for each $a \in A$ and $k \in R$ needs not be greater than $d_a^k x_a - h_a^k \nu$. Therefore, we can replace the constraint (15) for each $a \in A$ and $k \in R$ with $\omega_a^k = (d_a^k - h_a^k \nu) x_a$, which is equivalent to $\omega_a^k = \max\{d_a^k - h_a^k \nu, 0\} x_a$, since $x_a \in \mathbb{B}^m$ and $\omega_a^k \ge 0$ for all $a \in A, k \in R$. Now, consider the following set $F_r(\rho, \nu)$ for a given $\nu \in \mathbb{R}_+$ defined as

$$F_r(\rho, \nu) = \left\{ x \in X_{st} : \sum_{a\in A} \mu_a x_a + \sum_{a\in A} \sum_{k\in R} \max\left\{ d_a^k - h_a^k \nu, 0 \right\} x_a \le b - (\Phi^{-1}(\rho))^2 \nu \right\}.$$

Then, it is clear that $F_r(\rho) = \cup_{\nu\ge 0} F_r(\rho, \nu)$. Now, for given $\hat{x} \in F_r(\rho)$, suppose that we have the optimal extreme point solution $\hat{z}$ to $\beta(\rho, \hat{x}, r)$. Let $\hat{D} = \{(i,k) \in D : \hat{z}_{a(i)}^k > 0\}$, and let $(p, l) = \arg\min_{(i,k)\in\hat{D}}\{d_{a(i)}^k / h_{a(i)}^k :$

$\hat{z}^k_{a(i)} > 0\}$. Since $\beta(\rho, \hat{x}, r)$ is a linear program with one knapsack constraint (10), it can be readily verified that $\hat{z}^k_{a(i)} = 1$ for all $(i,k) \in \hat{D} \setminus \{(p,l)\}$ and $\hat{z}^l_{a(p)} = [(\Phi^{-1}(\rho))^2 - \sum_{(i,k)\in\hat{D}\setminus\{(p,l)\}} h^k_{a(i)}]/h^l_{a(p)}$. Let $\hat{q}$ be the index in $Q$ corresponding to $(p,l)$. Now, we set $\hat{\nu} = s_{\hat{q}}$. Then, for each pair of $i \in I_A$ and $r \in R$, we set $\hat{\omega}^k_{a(i)} = d^k_{a(i)} - h^k_{a(i)} s_{\hat{q}}$ if $(i,k) \in \hat{D}$, $\hat{\omega}^k_{a(i)} = 0$ otherwise. Then, it can be verified that $\hat{\nu}$ and $\hat{\omega}^k_a$ for all $a \in A, r \in R$ satisfy the constraints (15) - (17), and $\sum_{a\in A} \sum_{k\in R} d^k_a \hat{x}_a \hat{z}^k_a = (\Phi^{-1}(\rho))^2 \hat{\nu} + \sum_{a\in A} \sum_{k\in R} \hat{\omega}^k_a$. Therefore, if $\hat{x} \in F_r(\rho)$, then $\hat{x} \in F_r(\rho, s_{\hat{q}})$ for some $\hat{q} \in Q$, which means $F_r(\rho) \subseteq \cup_{q\in Q} F_r(\rho, s_q)$. Since $F_r(\rho) = \cup_{\nu\geq 0} F_r(\rho, \nu)$, $\cup_{q\in Q} F_r(\rho, s_q) \subseteq F_r(\rho)$, which means $F_r(\rho) = \cup_{q\in Q} F_r(\rho, s_q)$. Finally, observe that $F_r(\rho, s_q)$ for $q \in Q$ can be restated as

$$F_r(\rho, s_q) = \left\{ x \in X_{st} : \sum_{a\in A} \mu_a x_a + \sum_{(i,k)\in D_q} (d^k_{a(i)} - h^k_{a(i)} s_q) x_{a(i)} \leq b - (\Phi^{-1}(\rho))^2 s_q \right\}.$$

It means that we can determine if $F_r(\rho, s_q) \neq \emptyset$ for each $q \in Q$ by solving $\text{SP}_q$ and checking whether the optimal value is less than or equal to $b - (\Phi^{-1}(\rho))^2 s_q$. Therefore, $F_r(\rho) \neq \emptyset$ if and only if there exists $q \in Q$ such that the optimal objective value of $\text{SP}_q$ is less than or equal to $b - (\Phi^{-1}(\rho))^2 s_q$. □

## References


Ahuja, R. K., Mehlhorn, K., Orlin, J., & Tarjan, R. E. (1990). Faster algorithms for the shortest path problem. *Journal of the ACM (JACM)*, *37*, 213–223.

Azaron, A., & Kianfar, F. (2003). Dynamic shortest path in stochastic dynamic networks: Ship routing problem. *European Journal of Operational Research*, *144*, 138–156.

Beasley, J. E., & Christofides, N. (1989). An algorithm for the resource constrained shortest path problem. *Networks*, *19*, 379–394.

Bellman, R. (1958). On a routing problem. *Quarterly of Applied Mathematics*, *16*, 87–90.

Ben-Tal, A., & Nemirovski, A. (1999). Robust solutions of uncertain linear programs. *Operations Research Letters*, *25*, 1–13.

Chassein, A., Dokka, T., & Goerigk, M. (2019). Algorithms and uncertainty sets for data-driven robust shortest path problems. *European Journal of Operational Research*, *274*, 671–686.

Chassein, A. B., & Goerigk, M. (2015). A new bound for the midpoint solution in minmax regret optimization with an application to the robust shortest path problem. *European Journal of Operational Research*, *244*, 739–747.

Chen, B. Y., Shi, C., Zhang, J., Lam, W. H., Li, Q., & Xiang, S. (2017). Most reliable path-finding algorithm for maximizing on-time arrival probability. *Transportmetrica B: Transport Dynamics*, *5*, 248–264.

Cheng, J., & Lisser, A. (2015). Maximum probability shortest path problem. *Discrete Applied Mathematics*, *192*, 40–48.

Cherkassky, B. V., Goldberg, A. V., & Radzik, T. (1996). Shortest paths algorithms: Theory and experimental evaluation. *Mathematical Programming*, *73*, 129–174.

Conde, E. (2017). A minimum expected regret model for the shortest path problem with solution-dependent probability distributions. *Computers & Operations Research*, *77*, 11–19.

Dantzig, G. B. (1960). On the shortest route through a network. *Management Science*, *6*, 187–190.

Dijkstra, E. W. (1959). A note on two problems in connexion with graphs. *Numerische Mathematik*, *1*, 269–271.

Duque, D., & Medaglia, A. L. (2019). An exact method for a class of robust shortest path problems with scenarios. *Networks*, *74*, 360–373.

Fredman, M. L., & Tarjan, R. E. (1987). Fibonacci heaps and their uses in improved network optimization algorithms. *Journal of the ACM (JACM)*, *34*, 596–615.

Galbrun, E., Pelechrinis, K., & Terzi, E. (2016). Urban navigation beyond shortest route: The case of safe paths. *Information Systems*, *57*, 160–171.

Goldberg, N., & Poss, M. (2020). Maximum probabilistic all-or-nothing paths. *European Journal of Operational Research*, *283*, 279–289.

Guillot, M., & Stauffer, G. (2020). The stochastic shortest path problem: a polyhedral combinatorics perspective. *European Journal of Operational Research*, *285*, 148–158.

Halman, N., Kovalyov, M. Y., Quilliot, A., Shabtay, D., & Zofi, M. (2019). Bi-criteria path problem with minimum length and maximum survival probability. *OR Spectrum*, *41*, 469–489.

Han, J., Lee, K., Lee, C., Choi, K.-S., & Park, S. (2016). Robust optimization approach for a chance-constrained binary knapsack problem. *Mathematical Programming*, *157*, 277–296.

Henig, M. I. (1986). The shortest path problem with two objective functions. *European Journal of Operational Research*, *25*, 281–291.

Ishii, H., Shiode, S., Nishida, T., & Namasuya, Y. (1981). Stochastic spanning tree problem. *Discrete Applied Mathematics*, *3*, 263–273.

Johnson, D. S., & Garey, M. R. (1979). *Computers and intractability: A guide to the theory of NP-completeness*. WH Freeman.

Karp, R. M. (1972). Reducibility among combinatorial problems. In *Complexity of Computer Computations* (pp. 85–103). Springer.

Knauer, C, & Sommer, L., & Stehn, F. (2018). An FPTAS for an Elastic Shape Matching Problem with Cyclic Neighborhoods. In O. Gervasi et al. (Eds.), *Computational Science and Its Applications – ICCSA 2018. ICCSA 2018.* Springer, Cham. volume 10961 of *Lecture Notes in Computer Science*.

Korkmaz, T., & Krunz, M. (2003). Bandwidth-delay constrained path selection under inaccurate state information. *IEEE/ACM Transactions on Networking (ToN)*, *11*, 384–398.

Mirchandani, P. B. (1976). Shortest distance and reliability of probabilistic networks. *Computers & Operations Research*, *3*, 347–355.

Murthy, I., & Sarkar, S. (1997). Exact algorithms for the stochastic shortest path problem with a decreasing deadline utility function. *European Journal of Operational Research*, *103*, 209–229.

Nie, Y. M., & Wu, X. (2009). Shortest path problem considering on-time arrival probability. *Transportation Research Part B: Methodological*, *43*, 597–613.

Nikolova, E., Kelner, J. A., Brand, M., & Mitzenmacher, M. (2006). Stochastic shortest paths via quasi-convex maximization. In Y. Azar, & T. Erlebach (Eds.), *Algorithms – ESA 2006. ESA 2006.*. Springer, Berlin, Heidelberg. volume 4168 of *Lecture Notes in Computer Science*.

Nikolova, E. V. (2009). *Strategic algorithms*. Ph.D. thesis Massachusetts Institute of Technology.

Olya, M. H. et al. (2014). Applying dijkstra's algorithm for general shortest path problem with normal probability distribution arc length. *International Journal of Operational Research*, *21*, 143–154.

Raith, A., Schmidt, M., Schöbel, A., & Thom, L. (2018). Multi-objective minmax robust combinatorial optimization with cardinality-constrained uncertainty. *European Journal of Operational Research*, *267*, 628–642.

Ryu, J., & Park, S. (2021). Robust solutions for stochastic and distributionally robust chance-constrained binary knapsack problems. *arXiv preprint arXiv:2105.11875*.

Sigal, C. E., Pritsker, A. A. B., & Solberg, J. J. (1980). The stochastic shortest route problem. *Operations Research*, *28*, 1122–1129.

Taccari, L. (2016). Integer programming formulations for the elementary shortest path problem. *European Journal of Operational Research*, *252*, 122-130.

Wang, L., Yang, L., & Gao, Z. (2016). The constrained shortest path problem with stochastic correlated link travel times. *European Journal of Operational Research*, *255*, 43–57.

Wang, Z., You, K., Song, S., & Zhang, Y. (2020). Wasserstein distributionally robust shortest path problem. *European Journal of Operational Research*, *284*, 31–43.

Xiao, Y., Thulasiraman, K., Fang, X., Yang, D., & Xue, G. (2011). Computing a most probable delay constrained path: Np-hardness and approximation schemes. *IEEE Transactions on Computers*, *61*, 738–744.